Exercise 3.2.1

For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \leq x \leq L$). Compare $f(x)$ to its Fourier series:

(a) $f(x) = 1$

(b) $f(x) = x^2$

(c) $f(x) = 1 + x$

(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases}$

(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

Solution

The Fourier series expansion of $f(x)$, which is defined on $-L \leq x \leq L$ and assumed to be piecewise smooth, is a $2L$-periodic extension of this function over all of x. Where f is continuous, the expansion is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right),$$

and where there are jump discontinuities, the expansion takes the average value of f: $[f(x-) + f(x+)]/2$. The formulas for the coefficients are

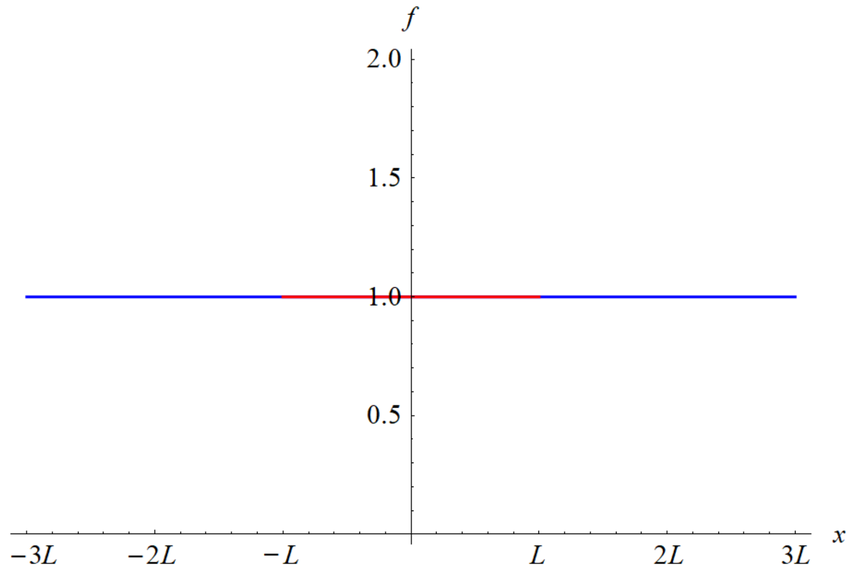
$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ A_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ B_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \end{aligned}$$

which are obtained by integrating both sides of the Fourier series expansion and taking advantage of the fact that the sine and cosine functions are orthogonal. In the following parts, $f(x)$ will be in red and the expansion will be in blue.

Part (a)

For $f(x) = 1$, the coefficients are

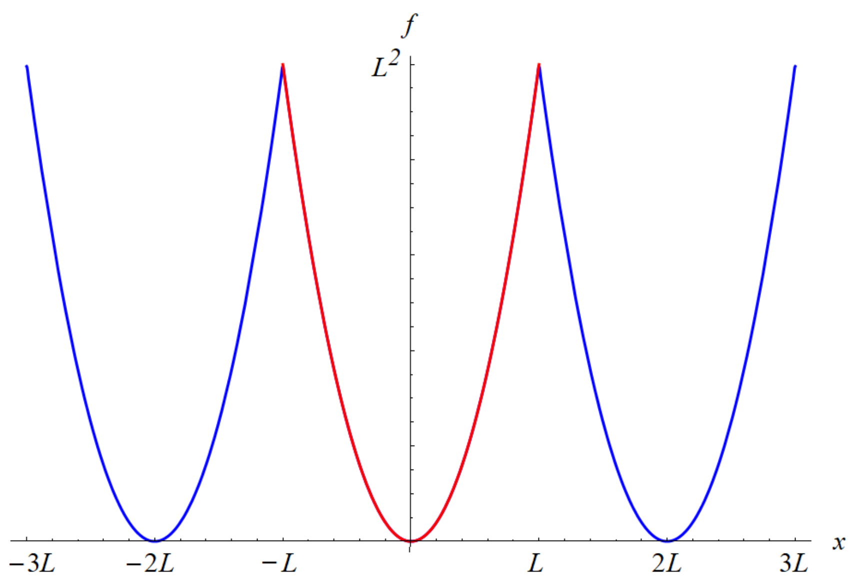
$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} dx = 1 \\ A_n &= \frac{1}{L} \int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx = 0 \\ B_n &= \frac{1}{L} \int_{-L}^{L} \sin \frac{n\pi x}{L}\, dx = 0. \end{aligned}$$



Part (b)

For $f(x) = x^2$, the coefficients are

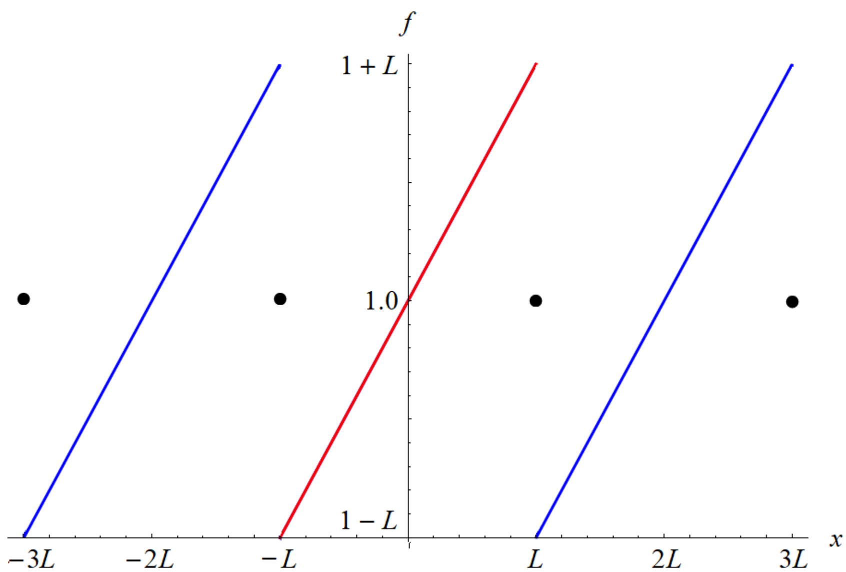
$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{L^2}{3}$$
$$A_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx = \frac{4(-1)^n L^2}{n^2\pi^2}$$
$$B_n = \frac{1}{L}\int_{-L}^{L} x^2 \sin\frac{n\pi x}{L}\,dx = 0.$$



Part (c)

For $f(x) = 1 + x$, the coefficients are

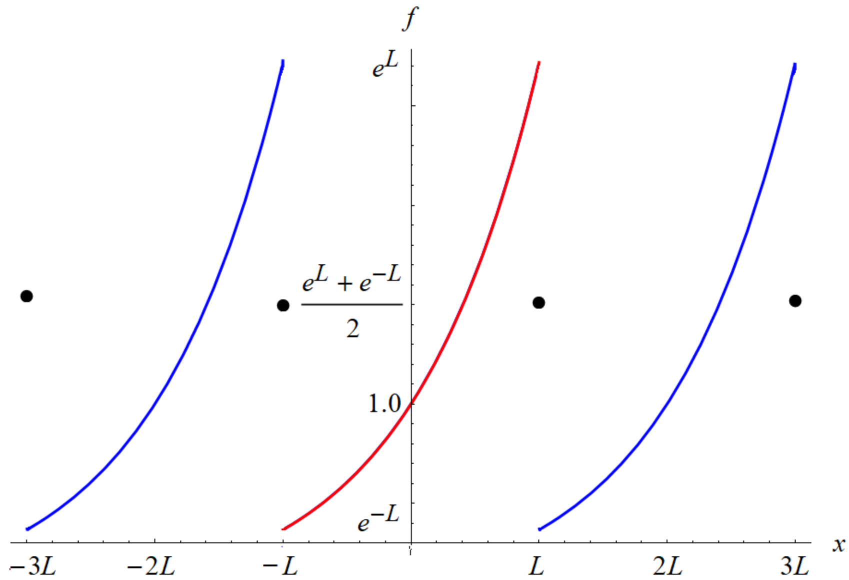
$$A_0 = \frac{1}{2L}\int_{-L}^{L} (1 + x)\, dx = 1$$
$$A_n = \frac{1}{L}\int_{-L}^{L} (1 + x)\cos\frac{n\pi x}{L}\, dx = 0$$
$$B_n = \frac{1}{L}\int_{-L}^{L} (1 + x)\sin\frac{n\pi x}{L}\, dx = -\frac{2(-1)^n L}{n\pi}.$$



Part (d)

For $f(x) = e^x$, the coefficients are

$$A_0 = \frac{1}{2L}\int_{-L}^{L} e^x\, dx = \frac{\sinh L}{L}$$
$$A_n = \frac{1}{L}\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\, dx = \frac{2(-1)^n L\sinh L}{n^2\pi^2 + L^2}$$
$$B_n = \frac{1}{L}\int_{-L}^{L} e^x\sin\frac{n\pi x}{L}\, dx = -\frac{2(-1)^n n\pi\sinh L}{n^2\pi^2 + L^2}.$$



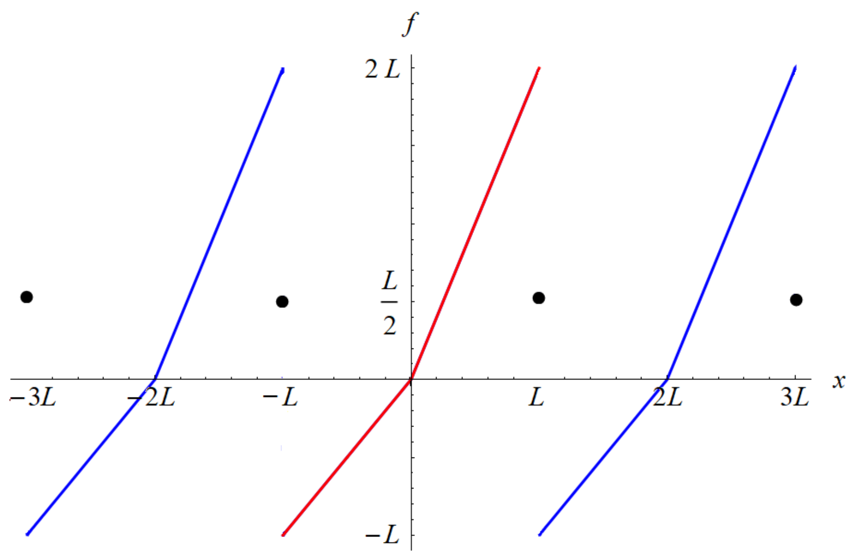
Part (e)

For $f(x) = x$ if $x < 0$ and $f(x) = 2x$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} x\,dx + \int_{0}^{L} 2x\,dx\right) = \frac{L}{4}$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{0} x\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} 2x\cos\frac{n\pi x}{L}\,dx\right) = \frac{[-1+(-1)^n]L}{n^2\pi^2}$$

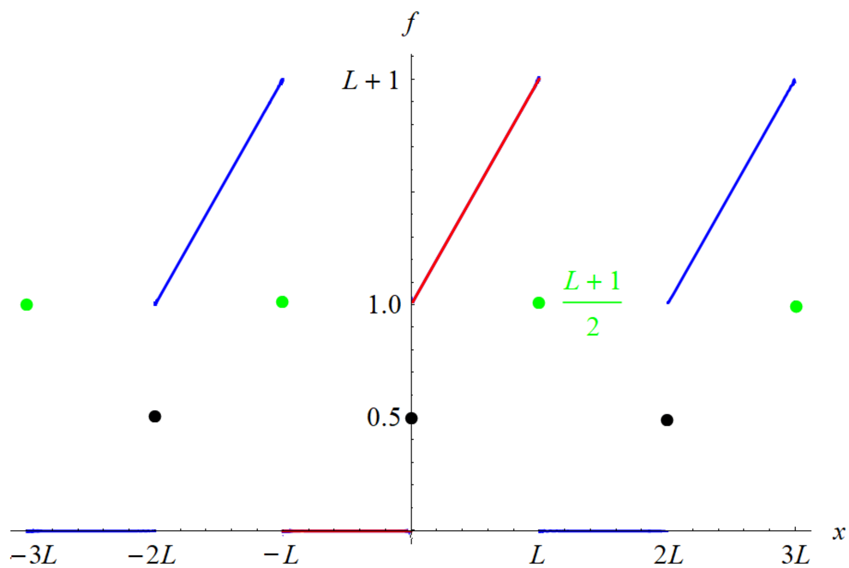
$$B_n = \frac{1}{L}\left(\int_{-L}^{0} x\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} 2x\sin\frac{n\pi x}{L}\,dx\right) = -\frac{3(-1)^n L}{n\pi}.$$



Part (f)

For $f(x) = 0$ if $x < 0$ and $f(x) = 1 + x$ if $x > 0$, the coefficients are

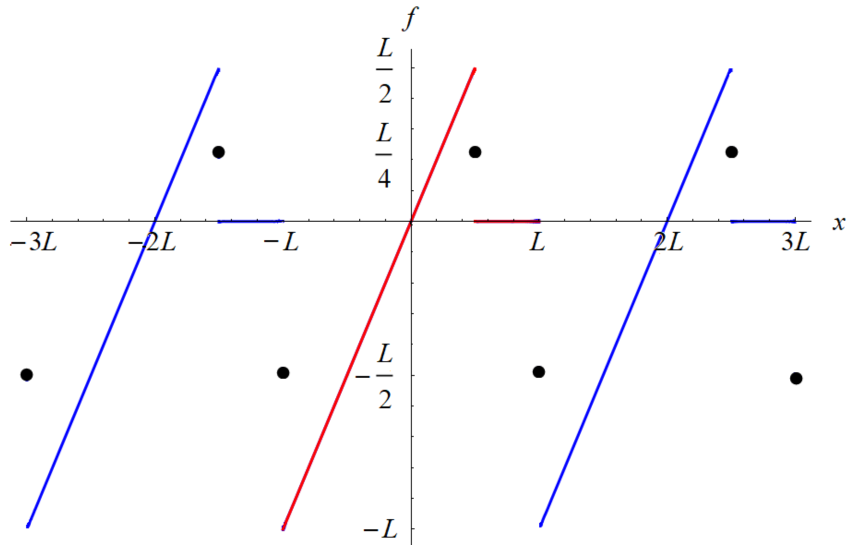
$$\begin{aligned}
A_0 &= \frac{1}{2L}\left[\int_{-L}^{0} 0\,dx + \int_0^L (1+x)\,dx\right] = \frac{L+2}{4} \\
A_n &= \frac{1}{L}\left[\int_{-L}^{0} 0\cos\frac{n\pi x}{L}\,dx + \int_0^L (1+x)\cos\frac{n\pi x}{L}\,dx\right] = \frac{[-1+(-1)^n]L}{n^2\pi^2} \\
B_n &= \frac{1}{L}\left[\int_{-L}^{0} 0\sin\frac{n\pi x}{L}\,dx + \int_0^L (1+x)\sin\frac{n\pi x}{L}\,dx\right] = \frac{1-(-1)^n(L+1)}{n\pi}.
\end{aligned}$$



Part (g)

For $f(x) = x$ if $x < L/2$ and $f(x) = 0$ if $x > L/2$, the coefficients are

$$\begin{aligned}
A_0 &= \frac{1}{2L}\left[\int_{-L}^{L/2} x\,dx + \int_{L/2}^{L} 0\,dx\right] = -\frac{3L}{16} \\
A_n &= \frac{1}{L}\left[\int_{-L}^{L/2} x\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\cos\frac{n\pi x}{L}\,dx\right] = \frac{L}{2n^2\pi^2}\left[-2(-1)^n + 2\cos\frac{n\pi}{2} + n\pi\sin\frac{n\pi}{2}\right] \\
B_n &= \frac{1}{L}\left[\int_{-L}^{L/2} x\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\sin\frac{n\pi x}{L}\,dx\right] = \frac{L}{2n^2\pi^2}\left[-2(-1)^n n\pi - n\pi\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2}\right].
\end{aligned}$$


f
L/2
L/4
x
−3L
−2L
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L
2L
3L
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